#### PhD Defense

Development of Optimized Operations for Homomorphic Encryption

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Manuscript: www.nicolasbon.com/phd

#### **Table of Content**

1 State of the Art
Fully Homomorphic Encryption
TFHE: FHE over the Torus
Bootstrapping and Negacyclicity

2 Contributions

Acceleration of Homomorphic Boolean Function Acceleration of large LUT Evaluation Transciphering with Transistor

3 Conclusion

# State of the Art

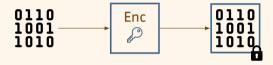
Part 1

State of the Art

Fully Homomorphic Encryption

#### Cryptography

#### Encryption:



#### Decryption:





#### Data at rest

- Hard Drives
- Cloud Storage





#### Data at rest

- Hard Drives
- Cloud Storage

#### Data in transit

- Internet Traffic
- Messaging / E-mails





- Hard Drives
- Cloud Storage



#### Data in transit

- Internet Traffic
- Messaging / E-mails





- Hard Drives
- Cloud Storage



#### Data in transit

- Internet Traffic
- Messaging / E-mails



#### Data at work

- Server-side computations
- AI-based services







#### Data at rest

- Hard Drives
- Cloud Storage

#### Data in transit

- Internet Traffic
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#### Data at work

- Server-side computations
- AI-based services

Questior

How to ensure secure computations?

Client



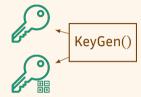
0110 1001 1010 Server







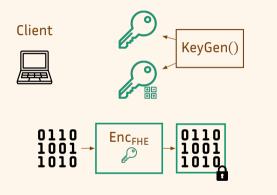




0110 1001 1010 Server



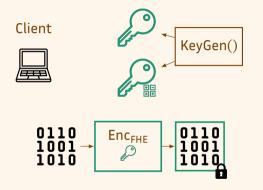




Server





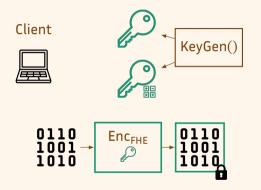




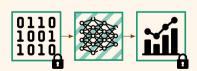


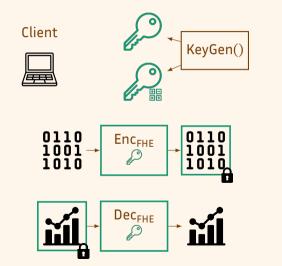






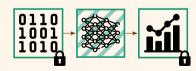




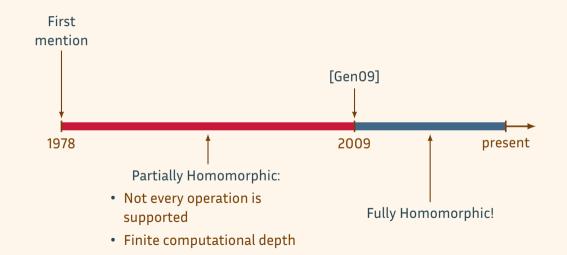


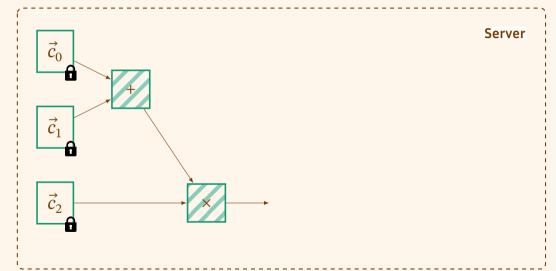
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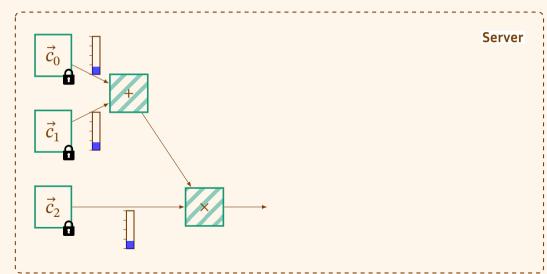


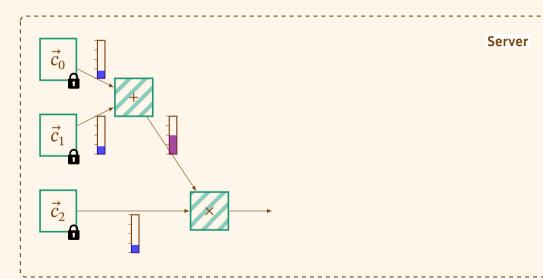


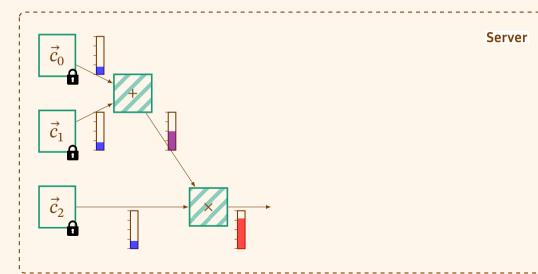
### **History of FHE**

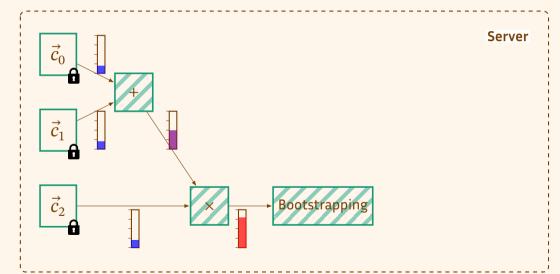


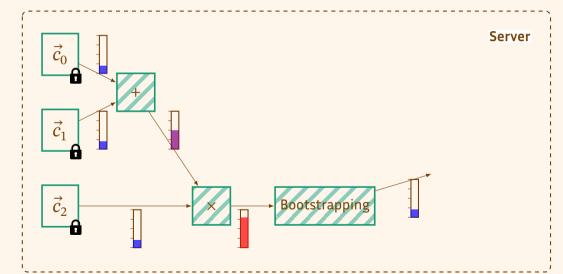


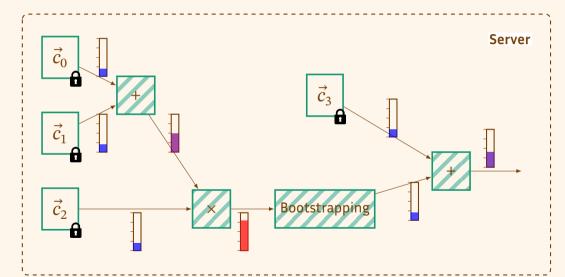


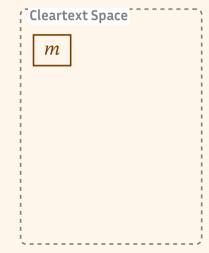


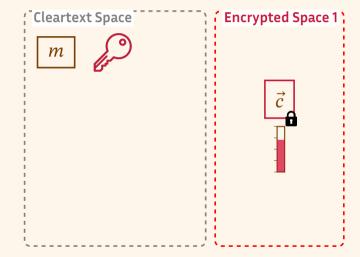


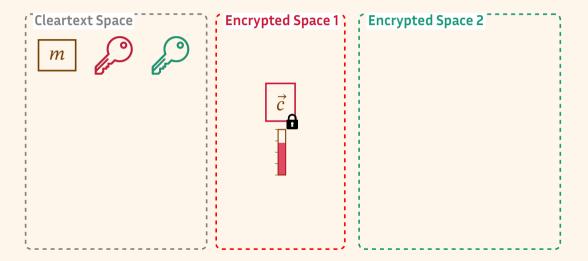


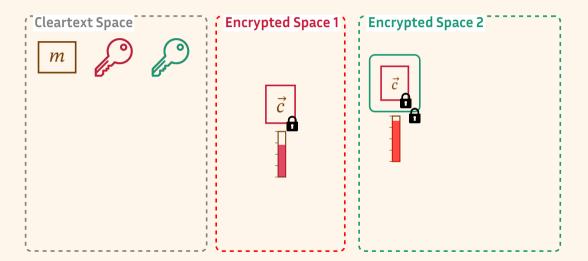


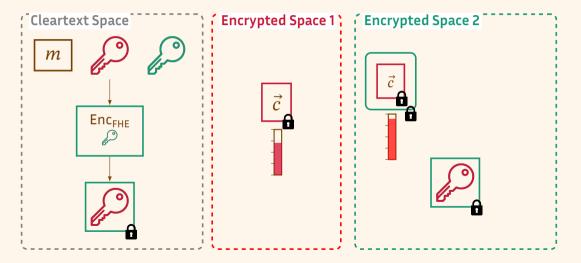


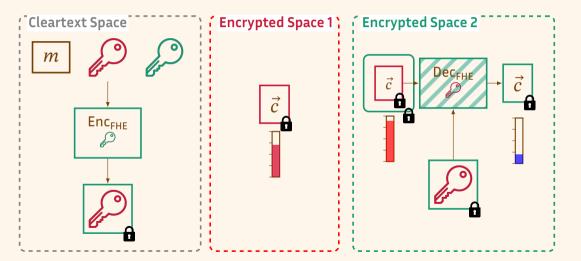










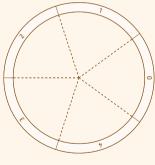


tate	of	the	Art

TFHE: FHE over the Torus

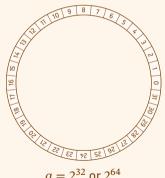
## **TFHE**: Description of the scheme

Clear Space:  $\mathbb{T}_p$ 



p has a size of a few bits.

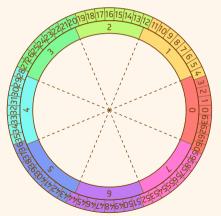
#### Encrypted Space: $\mathbb{T}_a$



 $q = 2^{32}$  or  $2^{64}$ .

## **TFHE:Description of the scheme**

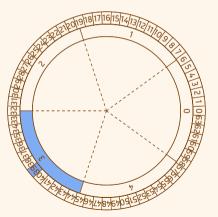
Natural Embedding of  $\mathbb{T}_p$  in  $\mathbb{T}_q$ 



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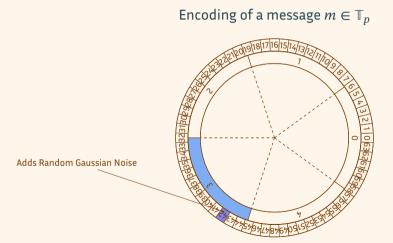
## **TFHE:Description of the scheme**

#### Encoding of a message $m \in \mathbb{T}_p$



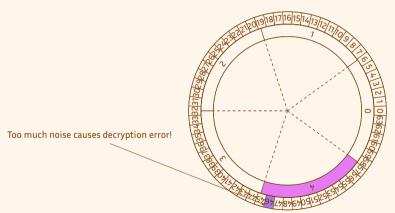
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# TFHE:Description of the scheme



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## Encoding of a message $m \in \mathbb{T}_p$



# Description of the scheme: Encryption Algorithm

#### Sampling of the secret key:

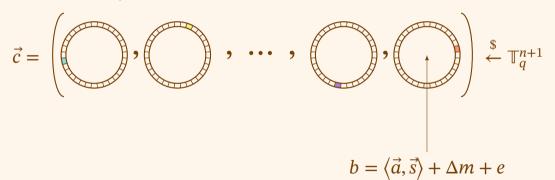
$$\vec{s} = (0, 1, \dots, 0) \stackrel{\$}{\leftarrow} \mathbb{B}^n$$

#### Sampling of a mask:

$$\vec{a} = \left( \begin{array}{c} \\ \\ \\ \end{array} \right), \quad \cdots , \quad \left( \begin{array}{c} \\ \\ \\ \end{array} \right) \stackrel{\$}{\leftarrow} \mathbb{T}$$

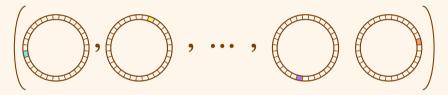
# Description of the scheme: Encryption Algorithm

#### Construction of ciphertext:



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#### Construction of ciphertext:

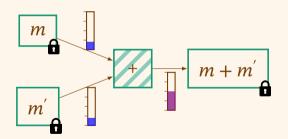


#### Decryption:

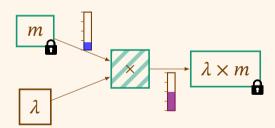
- 1. Recover the noisy message with  $\Delta m + e = b \langle \vec{a}, \vec{s} \rangle$ .
- 2. Round to the closest plaintext:  $m = \left\lfloor \frac{\Delta m + e}{\Delta} \right\rfloor$ .

## **Homomorphic Operations**

#### Additions of ciphertexts:

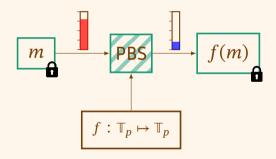


Multiplication of a ciphertext with a constant:



# **Programmable Bootstrapping**

Main Feature: the **Programmable** Bootstrapping



# Problem: PBS is very slow in large spaces

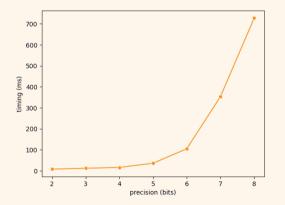


Figure: Degradation of the timing of a PBS, with respect to  $\log_2(p)$ .

**Bootstrapping and Negacyclicity** 

Decryption procedure is done is two steps:

- Computing  $b \langle \vec{a}, \vec{s} \rangle = m + e$
- Rounding to the closest plaintext:  $\lfloor m + e \rfloor = m$ .

Decryption procedure is done is two steps:

- Computing  $b \langle \vec{a}, \vec{s} \rangle = m + e \rightarrow \textbf{Easy}$
- Rounding to the closest plaintext:  $\lfloor m + e \rfloor = m$ .

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- Rounding to the closest plaintext: [m + e] = m. -> More challenging

Decryption procedure is done is two steps:

- Computing  $b \langle \vec{a}, \vec{s} \rangle = m + e \rightarrow Easy$
- Rounding to the closest plaintext: [m + e] = m. -> More challenging

#### **Ouestion**

How to perform rounding homomorphically?

Works in the polynomial ring  $\mathbb{Z}[X]/(X^N+1)$  (with N a power of two).

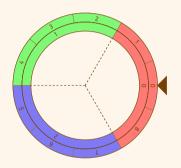
Works in the polynomial ring  $\mathbb{Z}[X]/(X^N+1)$  (with N a power of two).

Let 
$$v(X) = v_0 + v_1 \cdot X + \dots + v_{N-1} X^{N-1}$$
 and  $a \in \mathbb{Z}_{2N}$ .

In this ring, something interesting happens:

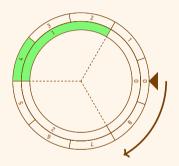
$$X^{-a} \cdot v(X) = \begin{cases} v_a + v_{a+1} \cdot X + \dots & \text{if } i \in [0, N[.\\ -v_a - v_{a+1} \cdot X + \dots & \text{if } i \in [N, 2N[.]] \end{cases}.$$

Blind rotation allows to homomorphically remove the noise:



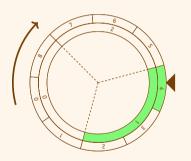
$$v_0 + v_1 X + v_2 X^2 + v_3 X^3 + v_4 X^4 + v_5 X^5 + v_6 X^6 + v_7 X^7 + v_8 X^8$$

Blind rotation allows to homomorphically remove the noise:



$$X^{-4} \cdot (v_0 + v_1 X + v_2 X^2 + v_3 X^3 + v_4 X^4 + v_5 X^5 + v_6 X^6 + v_7 X^7 + v_8 X^8)$$

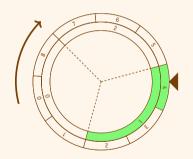
Blind rotation allows to homomorphically remove the noise:



$$v_4 + v_5 X + v_6 X^2 + v_7 X^3 + v_8 X^4 - v_0 X^5 - v_1 X^6 - v_2 X^7 - v_3 X^8$$

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Blind rotation allows to homomorphically remove the noise:



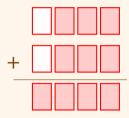
 $f(1) + f(2)X + f(2)X^{2} + f(2)X^{3} - f(0)X^{4} - f(0)X^{5} - f(0)X^{6} - f(1)X^{7} - f(1)X^{8}$ 

- Taking q=2N is unrealistic, so we modswitch the ciphertext from  $\mathbb{Z}_q$  to  $\mathbb{Z}_{2N}$ .
- If the message is encoded lies between N and 2N, then the output will be wrong.

#### Bit of Padding:

Pragmatic solution: enforce the MSB of ciphertexts to zero.

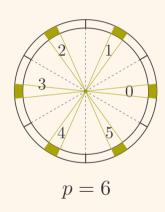
# The bit-of-padding problem



#### Problem

The bit-of-padding technique prevents from using the linear homomorphism!

# **Parity**





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# **Parity**

New Accumulator:



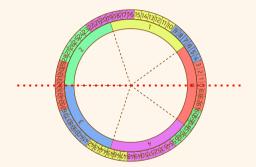
#### **Odd-Modulus Accumulator:**



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# **Parity**

New Accumulator:



Takeaway

Odd moduli naturally solve the negacyclicity problem!

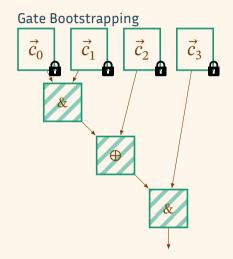
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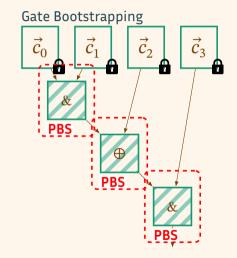
Contributions

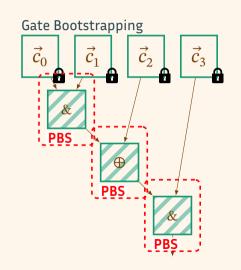
Part 2

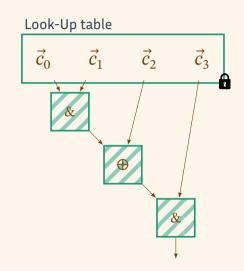
# Contributions Acceleration of Homomorphic Boolean

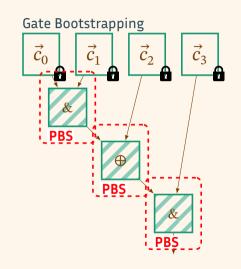
**Function** 

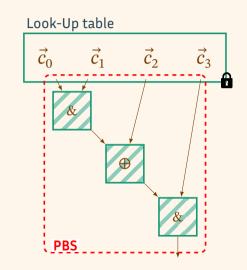








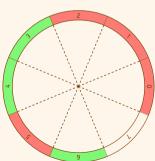




# The *p*-encodings

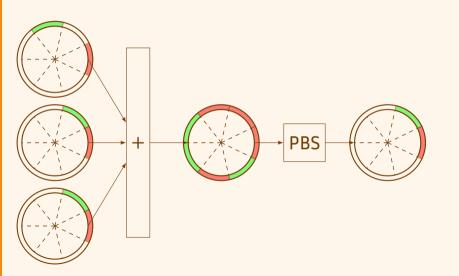
A *p*-encoding is a function  $\mathcal{E}: \mathbb{B}: \mapsto 2^{\mathbb{Z}_p}$ .

$$\mathcal{E} : \begin{cases} 0 \mapsto \{\alpha_i\}_{0 \le i \le l_0} \\ 1 \mapsto \{\beta_i\}_{0 \le i \le l_1} \end{cases}$$



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# **New Function Evaluation Algorithm**



$b_1$	$b_2$	$b_3$	f
0	0	0	0
1	0	0	1
0	1	0	0
:	:	÷	:

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# Advantages of the method

- One single bootstrapping to evaluate the whole function
- Plaintext space significantly smaller than  $2^{\ell}$

Takeaway:

Better scaling than the traditional approaches

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# Boolean function on p-encodings

#### **Ouestion**

For a given function  $f: \mathbb{B}^{\ell} \to \mathbb{B}$ , how to find a valid set of p-encodings (and the best p)?

# Boolean function on p-encodings

#### **Ouestion**

For a given function  $f: \mathbb{B}^{\ell} \to \mathbb{B}$ , how to find a valid set of p-encodings (and the best p)?

Exhaustive search is too costly when  $\ell$  grows.

#### Reduction of the search space

If a solution exists, then it can be reduced to the form:

$$\mathcal{E}_i = \begin{cases} 0 \mapsto \{0\} \\ 1 \mapsto \{d_i\} \end{cases} \quad \text{with: } \mathcal{E}_0 = \begin{cases} 0 \mapsto \{0\} \\ 1 \mapsto \{1\} \end{cases}$$

#### Another point of view on the problem

#### Truth table of *f*:

$b_1$	$b_2$	$b_3$	$f(b_1, b_2, b_3)$
0	0	0	0
1	0	0	1
0	1	0	0
:	:	:	:

$$\begin{cases} 0 \cdot d_1 + 0 \cdot d_2 + 0 \cdot d_3 = r_0 \\ 1 \cdot d_1 + 0 \cdot d_2 + 0 \cdot d_3 = r_1 \\ 0 \cdot d_1 + 1 \cdot d_2 + 0 \cdot d_3 = r_2 \\ & \vdots \end{cases}$$

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#### Another point of view on the problem

#### Truth table of *f*:

$b_1$	$b_2$	$b_3$	$f(b_1, b_2, b_3)$		
0	0	0	0		
1	0	0	1		
0	1	0	0		
:	:	:	:		

$$\begin{cases} 0 \cdot d_1 + 0 \cdot d_2 + 0 \cdot d_3 &= r_0 \\ 1 \cdot d_1 + 0 \cdot d_2 + 0 \cdot d_3 &= r_1 \\ 0 \cdot d_1 + 1 \cdot d_2 + 0 \cdot d_3 &= r_2 \\ &\vdots \end{cases}$$

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### Another point of view on the problem

By writing all the inequations  $\neq$  we get:

```
\begin{cases} c_1^{(1)}d_1 + \dots + c_\ell^{(1)}d_\ell \not\equiv 0 \pmod{p}, \\ c_1^{(2)}d_1 + \dots + c_\ell^{(2)}d_\ell \not\equiv 0 \pmod{p}, \\ \vdots \\ c_1^{(2^{\ell-1})}d_1 + \dots + c_\ell^{(2^{\ell-1})}d_\ell \not\equiv 0 \pmod{p}. \end{cases}
```

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#### The search algorithm

- Our search algorithm finds a solution for a given p.
- To identify relevant values for p, we developed a **heuristic** method that finds an upper bound on the optimal p.

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# **Experimental Results**

Primitive	Implementation	Performances
One full run of SIMON	Gate Bootstrapping	174 s
	[BSS <sup>+</sup> 23]	128 s
	Our work	10 s
One warm-up phase of Trivium	Gate Bootstrapping	1498 s
	[BOS23] (estimation on our machine)	53 s
	Our work	32.8 s
One Full Keccak permutation	Gate Bootstrapping	30.7 min
	Our work	8.8 min
One Ascon hashing	Gate Bootstrapping	200s
	Our work	92 s

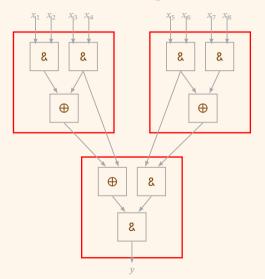
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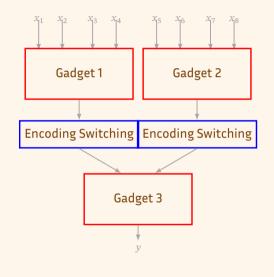
# **Experimental Results**

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One full run of SIMON	Gate Bootstrapping	174 s
p=9	[BSS <sup>+</sup> 23]	128 s
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One warm-up phase of Trivium	Gate Bootstrapping	1498 s
	[BOS23] (estimation on our machine)	53 s
p = 9	Our work	32.8 s
One Full Keccak permutation	Gate Bootstrapping	30.7 min
p=3	Our work	8.8 min
One Ascon hashing	Gate Bootstrapping	200s
p = 17	Our work	92 s

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## Extension to larger circuits





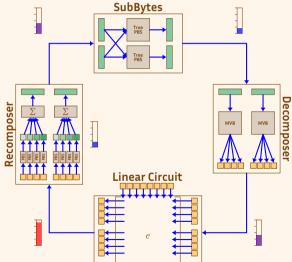
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### **Performances of AES**

One full evaluation of AES-128	[GHS12] †	18 min
$(\epsilon = 2^{-23})$ on one thread	[CLT14] †	5 min
p = 17	[TCBS23]	270 s
p = 17	Our work	103 s
One full evaluation of AES-128	Gate Bootstrapping	234 s
( $\epsilon=2^{-40}$ ) on one thread	Our work (Real implementation)	135 s
p = 17	Our work (Theoretical timing with two keys)	105 s

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# Hippogryph: a better version of homomorphic AES



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celeration	of large LUT	Evaluatio	n

Contributions

#### **Evaluation of large LUT**

PBS is only efficient at small precision, so we cannot evaluate large LUT directly with it.

Question

How to decompose a LUT into small PBS operations?

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#### An analogy with side-channels protection

- To protect the evaluation of LUT (e.g. S-box of AES) against side-channel attacks, masking is usually used.
- · Masking AND gates is the most costly.
- Techniques to generate masked circuits minimizing the number of AND gates.

#### **Ouestion**

Can we do the same, but minimizing the number of PBS calls?

## Formalization of the problem

$$f: \mathbb{Z}_t \mapsto \mathbb{Z}_{t'}$$

Taking  $t = s^n$ , we manipulate blocks of size s. If s is not prime, we encode them in  $\mathbb{F}_p$ .

$$\mathbb{Z}_s \subseteq \mathbb{F}_p$$

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#### Generation of a decomposition

Construction of a pool of derived random variables:

$$\forall j \in \{n, \lambda - 1\}, x_j = \phi_j(x_0, \dots, x_{j-1}) = \psi_j\left(\sum_{k=0}^{j-1} \alpha_k \cdot x_k\right)$$

Layout of the decomposition:

$$f(x_0,\ldots,x_{n-1}) = \sum_{i=0}^{t-1} \left(\sum_{j=0}^{n+\lambda-1} \beta_{i,j} \cdot x_j\right) \cdot \left(\sum_{k=0}^{n+\lambda-1} d_{i,k} \cdot x_j\right)$$

- $d_{i,k}$ : generated at random
- $\beta_{i,j}$ : determined by the algorithm

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### Cost of a decomposition

Homomorphic multiplications can be done with **two** calls of the PBS with function  $x \mapsto x^2$ .

It comes from the identity:  $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$ .

#### Cost of a decomposition

$$\#_{PBS} = \lambda + 2t$$

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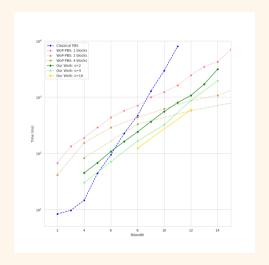
#### Determinations of the $\beta$ 's

$$s^{n} \left\{ \left( \begin{array}{c} y_{0} \\ y_{1} \\ \vdots \\ \vdots \\ y_{s^{n}-1} \end{array} \right) = \left( \begin{array}{c|c} t \cdot (n+\lambda) \\ \hline \mathcal{A}_{0} \\ \hline \end{array} \right) \cdot \left( \begin{array}{c} \beta_{0,0} \\ \vdots \\ \hline \beta_{0,n+\lambda-1} \\ \hline \vdots \\ \hline \beta_{t-1,0} \\ \vdots \\ \hline \beta_{t-1,n+\lambda-1} \end{array} \right)$$

$$\mathcal{A}_{i} = \begin{pmatrix} x_{0}^{(0)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(0)} \right\rangle & \dots & x_{n+\lambda-1}^{(0)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(0)} \right\rangle \\ x_{0}^{(1)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(1)} \right\rangle & \dots & x_{n+\lambda-1}^{(1)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(1)} \right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ x_{0}^{(s^{n}-1)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(s^{n}-1)} \right\rangle & \dots & x_{n+\lambda-1}^{(s^{n}-1)} \cdot \left\langle \vec{d}_{i}, \vec{x}^{(s^{n}-1)} \right\rangle \end{pmatrix}$$

PhD Defense 38/46

## **Performances**



PhD Defense 39/46

Transcipheri	ng with Tra	nsistor	

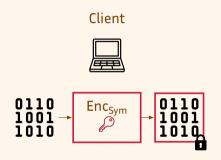
Contributions

Client



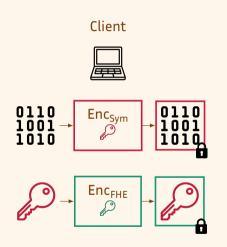
Server





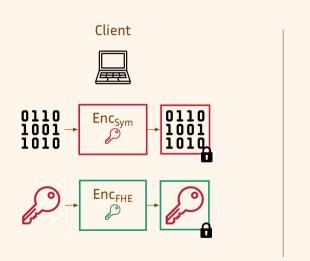
Server

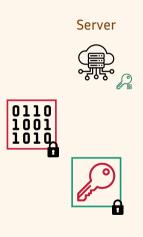


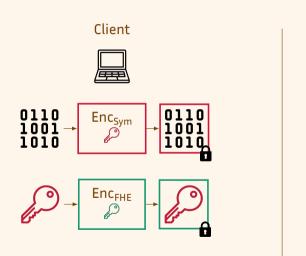


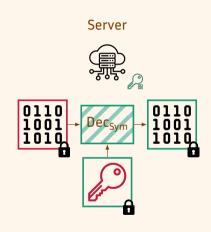
Server

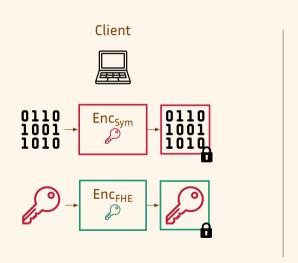


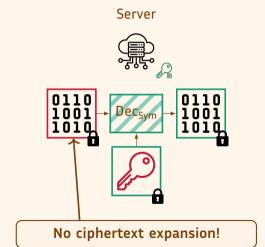








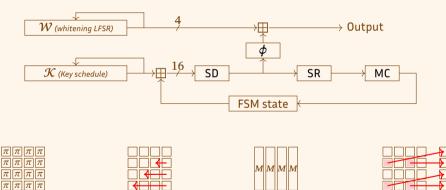




We propose a new symmetric cipher, that shows good performances in the homomorphic domain.

Prime field: F<sub>17</sub>

(a) SD.



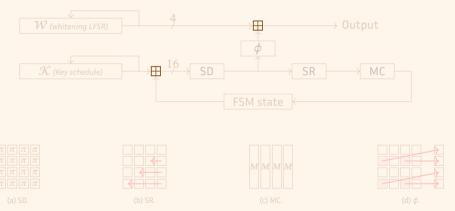
PhD Defense 41/46

(c) MC.

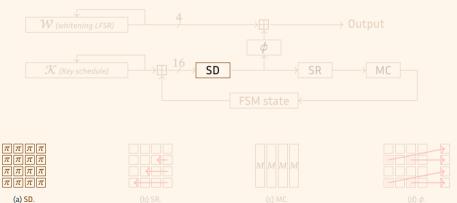
(d)  $\phi$ .

(b) SR.

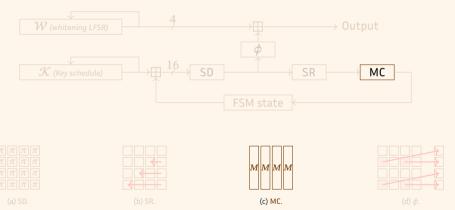
#### Prime field: $\mathbb{F}_{17}$



#### Prime field: $\mathbb{F}_{17}$



#### Prime field: $\mathbb{F}_{17}$



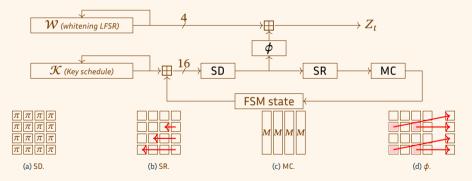
#### **MixColumns**

The matrix we chose for MixColumns is:

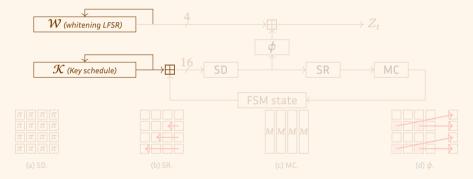
$$M = \left[ \begin{array}{cccc} 2 & 1 & 1 & 1 \\ 1 & -1 & 1 & -2 \\ 1 & 1 & -2 & -1 \\ 1 & -2 & -1 & 1 \end{array} \right].$$

- · Matrix MDS to ensure optimal diffusion,
- · Symmetric,
- Minimal  $\ell_2$ -norm of 7  $\rightarrow$  important for noise management.

Prime field: F<sub>17</sub>

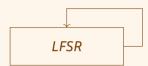


Prime field:  $\mathbb{F}_{17}$ 





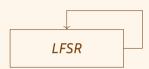
• Naive approach: linearly update the state at each clock.



- Naive approach: linearly update the state at each clock.
- Problem: the noise accumulates over time.



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- Problem: the noise accumulates over time.
- Solution: Computing on the fly the coefficients of the linear combination in clear



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- Problem: the noise accumulates over time.
- · Solution: Computing on the fly the coefficients of the linear combination in clear

The noise variance in the output of the silent LFSR remains stable over time, without using any PBS.

#### **Performances**

Cipher	Setup	Latency	Throughput	Communication Cost <sup>a</sup>	$p_{err}$
Trivium [BOS23] (128 thr.)	2259 ms	121 ms	529 bits/s	640 B + 35.6 MB <sup>†</sup>	2-40
Kreyvium [BOS23] (128 thr.)	2883 ms	150 ms	427 bits/s	1024 B + 35.6 MB <sup>†</sup>	2-40
Margrethe [HMS23]	No	27.2 ms	147.06 bits/s	64 MB *	$< 2^{-1000}$
Hargrettle [HM323]	No	54.2 ms	73.8 bits/s	128 MB *	$< 2^{-1000}$
PRF-based construction [DJL <sup>+</sup> 24]	No	5.675 ms	881 bits/s	32.8 MB = 8.9 MB + 23.9 MB	$2^{-64}$
FRAST [CCH+24]	25 s (8 thr.)	6.2 s	20.66 bits/s	34.05 MB = 148 KB + 33.91 MB	2-80
Transistor	No	251 ms	65.10 bits/s	13.54 MB = 780 B + 12.78 MB	$2^{-128}$

<sup>&</sup>lt;sup>a</sup> Includes size of encrypted symmetric key + size of evaluation keys. † Values recomputed from the data of the papers. For consistency's sake, we applied the classical technique of ciphertexts compression to estimate the communication cost.

<sup>\*</sup> In Margrethe, no keyswitching nor bootstrapping keys are required.

# Conclusion

Part 3

#### Conclusion

Studying plaintext spaces not power of two yielded speed-ups in different use-cases:

- Acceleration of boolean functions
- Acceleration of large LUT
- · Improvement of transciphering performances

#### Perspectives

- More study of those "exotic" plaintext spaces
- · Can we do this kind of things with packed schemes such as CKKS?

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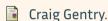
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